METHOD FOR SOLVING THE STEADY-STATE PROBLEM OF RADIATIVE-CONDUCTIVE HEAT TRANSFER IN A PLANE SLAB. II

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An algorithm is described for solving the problem of radiative-conductive heat transfer in a plane slab of a nongray medium with opaque mirror boundaries. The influence of the total energy flux on the heat transfer is studied.

The numerical implementation of the method described in the first part of this study [1] is essentially the reduction of the integrals in Eqs. (16)-(19) to quadratures.* This is not a trivial procedure, because of the integrals over the wavelength and, especially, along the coordinates.

With regard to the integration over the wavelength, we note that it is not desirable to use a quadrature equation immediately over the entire spectrum. A much better accuracy, with the same number of calculation points, can be achieved by partitioning the integration range into subranges, within each of which there is a corresponding average density of calculation points. This approach is based on both the behavior of the spectral characteristics (primarily k_ν) and the distance from the maximum of the function $I^1_{p,\nu}$.

With regard to the integration along the coordinate, we note that a difficulty arises here because of the functions $e^{-\gamma_\nu(2\pm\xi\pm x)}$ and $e^{-\gamma_\nu|x-\xi|}$ which appear in the expression for the Green's function. At large values of γ_ν these functions as well as their derivatives with respect to ξ change very rapidly near the points $x = \xi = \mp 1$ and $\xi = x$, respectively. Although large values of γ_ν are reached at either large values of τ_ν or small values of μ, i.e., when the directed spectral fluxes make a small contribution to the total energy flux, we need an adequate number of calculation points l_x in order to achieve accurate results. This approach runs into a considerable cost in computer time, since the time required is roughly proportional to l_x^2.

In this connection we take special measures to substantially reduce the number of calculation points along the coordinate, at no loss of accuracy. The singularities due to the behavior of the functions $e^{-\gamma_\nu(2\pm\xi\pm x)}$ are taken into account through the partitioning of the interval $[-1, 1]$ into subintervals, within each of which there is a corresponding average density of calculation points. This average density increases rapidly for the subintervals near the points $\xi = \pm 1$. This procedure is particularly effective for radiative-conductive heat transfer with a small value of N, a moderate value of τ, and a large value of R, i.e., in the case in which the temperature and its derivative vary rapidly near the boundaries.

We also single out the singularities due to the behavior of $e^{-\gamma_\nu|x-\xi|}$ and due to the discontinuity of the derivative of the Green's function (but not of its primitive) at $\xi = x$, through a breaking up of the kernels of the integral equations into parts, one of which is a smooth function of ξ, while the other contains all the singularities and can be calculated analytically. We first consider the kernel which contains the primitive of the Green's function. We write it as $j_\nu = j_{1,\nu} + j_{2,\nu}$, where

$$j_{2,\nu} = \operatorname{sign}(x-\xi)\,(\beta_\nu/\gamma_\nu)\,(e^{-\gamma_\nu|x-\xi|}-1). \tag{22}$$

The function $j_{2,\nu}$ contains all the singularities due to the behavior of $e^{-\gamma_\nu|x-\xi|}$ so that $j_{1,\nu}$ is smooth. We can therefore write the last term on the right side of Eq. (16) as

$$\hat{A}\theta = \sum_i \left\{ \int_{-1}^{1} [J_i(x,\xi)\,\hat{\eta}_i\theta(\xi) - J_{2,i}(x,\xi)\,\hat{\eta}_i\theta(x)]\,d\xi + V_i(x)\,\hat{\eta}_i\theta(x) \right\}, \tag{23}$$

*The sequence of equation numbers in this paper is a continuation of that in [1].

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where

$$J_{2,i}(x,\ \xi) = \int_0^\infty \int_0^1 \tau_\nu n_\nu^2 I_{p,\nu}^{(i)}(T_0)\, j_{2,\nu}(x,\ \xi)\, d\nu d\mu, \tag{24}$$

and V_i is found by integrating $j_{2,\nu}$:

$$V_i(x) = \int_0^\infty \int_0^1 \tau_\nu n_\nu^2 I_{p,\nu}^{(i)}(T_0)\,(\beta_\nu/\gamma_\nu)\,\{[e^{-\gamma_\nu(1-x)} - e^{-\gamma_\nu(1+x)}]/\gamma_\nu - 2x\}\, d\nu d\mu. \tag{25}$$

The elimination of singularities from the kernel of Eq. (17) is slightly more complicated, since here we run into the additional difficulty posed by the discontinuity of the derivative of the Green's function. Writing g_ν in the form $g_\nu = g_{1,\nu} + g_{2,\nu}$, where the function $g_{2,\nu}$ carries all the singularities,

$$g_{2,\nu} = -\beta_\nu e^{-\gamma_\nu|x-\xi|} + |x-\xi|\,(1/2 - \gamma_\nu\beta_\nu), \tag{26}$$

we find the following representation for the integral term in Eq. (17):

$$\hat{B}\theta = \sum_i \left\{ \int_{-1}^{1} [G_i(x,\ \xi)\,\hat{\eta}_i\theta(\xi) - G_{2,i}(x,\ \xi)\,\hat{\eta}_i\theta(x)]\, d\xi + W_i(x)\,\hat{\eta}_i\theta(x) \right\}. \tag{27}$$

Here

$$G_{2,i}(x,\ \xi) = \int_0^\infty \int_0^1 \tau_\nu n_\nu^2 I_{p,\nu}^{(i)}(T_0)\, g_{2,\nu}(x,\ \xi)\, d\nu d\mu, \tag{28}$$

and W_i is the result of the integration of $g_{2,\nu}$,

$$W_i(x) = \int_0^\infty \int_0^1 \tau_\nu n_\nu^2 I_{p,\nu}^{(i)}(T_0) \left\{ \frac{\beta_\nu}{\gamma_\nu}\,[e^{-\gamma_\nu(1+x)} + e^{-\gamma_\nu(1-x)} - 2] + (1+x^2)\left(\frac{1}{2} - \gamma_\nu\beta_\nu\right) \right\} d\nu d\mu. \tag{29}$$

Accordingly, after these manipulations, the integrands in (23) and (27) are functions which, along with their first derivatives, have no singularities due to the functions $e^{-\gamma_\nu|x-\xi|}$. The calculations show that serious errors result from a neglect of the singularities, especially at small values of the parameter N and at values of τ on the order of a few units. For example, with N = 0.2, τ = 5, R = 0 and a = 0.05, and with l_x = 14 calculation points, calculations neglecting the singularities lead to values $\Delta\theta = 1 - \theta(1)$, different from the actual values by a factor of about two.

A variety of iteration methods can be used for a numerical solution of Eqs. (16) and (17). In this particular case it is essentially inconsequential which method is used, as long as the convergence is sufficiently rapid. Most of the computer time is expended on evaluating the kernels of the integral equations. These kernels are stored in an external memory and then recovered at each iteration step as necessary. It is thus possible to simultaneously solve several problems of radiative–conductive heat transfer, if they are all governed by the same kernels. For "gray" problems with m = 4, each series actually corresponds to a certain value of the reflection coefficient and to a certain optical thickness, while within the series the values of the thermal conductivity and the total flux can be completely arbitrary.

In this regard the situation is much worse for "selective" problems. For the given spectrum (n_ν, k_ν, R_ν), the governing parameter of the series (in addition to h) is T_0. This temperature cannot be chosen to be the same for all types of radiative–conductive heat transfer with equal values of T_1, N, and a, since an adequate accuracy is achieved through the appropriate choice of T_0, because the method can be implemented effectively only for values of m on the order of a few units. However, this situation does not prevent us from using the same value of T_0 and thus the same kernels in "selective" problems with different values of T_1, N, and a if the temperature distributions corresponding to these regimes are not greatly different.

In the present study we use the following iteration process:

$$\theta^{(n+1)} = \alpha\theta^{(n)} + (1-\alpha)\,[A\hat{\theta}^{(n)} + 1 - (a/N)(1+x)], \tag{30}$$

$$\theta'^{(n+1)} = \alpha\theta'^{(n)} + (1-\alpha)\,(\hat{B}\theta^{(n)} - a/N), \tag{31}$$

where the parameter α ($0 \le \alpha < 1$) regulates the convergence rate. In problems with small values of R, τ, and, especially, N and with large values of a, the convergence is achieved with values of α of about 1. With $N \geqslant 4$ and $\alpha = 0$, convergence is achieved for any values of R, τ, and a, i.e., a simple iteration method is difficult.

TABLE 1. Comparison with the Results of Numerical Calculations of Other Investigators

Data of [2-5]		τ=0.05	τ=0.5	τ=5
[2] (R=0; N=0,8; Δθ=0,5)		a=1,0799 Δθ=0,5000	a=0,7694 Δθ=0,5000	a=0,315 Δθ=0,5001
[3] (N=0,8; Δθ=0,9)	R=0,9	a=0,438 Δθ=0,8984	a=0,559 Δθ=0,8972	a=0,466 Δθ=0,8971
	n_1=25 $\varkappa_1$=23,43		a=0,556* Δθ=0,8987	
[4] (R=0,9; N=0,8; Δθ=0,9)			a=0,56004 Δθ=0,9001	
[5] (R=0; N=0,4181; Δθ=0,0756)			a=0,1738 Δθ=0,07469	a=0,05372 Δθ=0,08438

*Data obtained with an account of the angular dependence of the reflection coefficient R_ν which is determined on the basis of the Fresnel equations in terms of n_1 and $\varkappa_1$ (the refractive index of the medium is 1). The hemispherical reflectivity is 0.9.

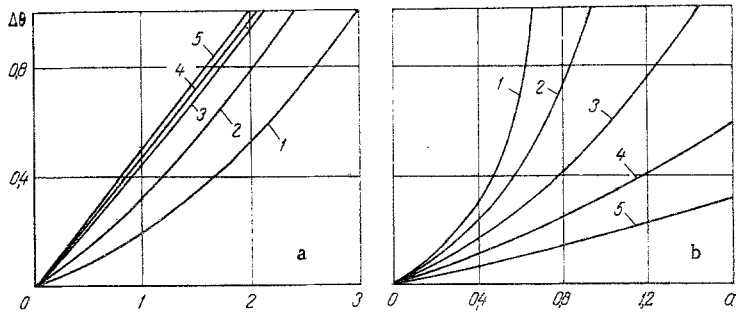

Fig. 1. Temperature drop in the plane slab, $\Delta\theta$, as a function of the total energy flux density. a: R = 0, N = 4. 1) τ = 0; 2) 1; 3) 5; 4) 10; 5) ∞; b: R = 0, τ = 1. 1) N = 0.5; 2) 1; 3) 2; 4) 4; 5) 8.

As the zeroth approximation we choose the solution of the linearized problem. The calculations are stopped under the condition

$$\max_j \left| \frac{(d\theta/dx)_j^{(n+1)} - (d\theta/dx)_j^{(n)}}{(d\theta/dx)_j^{(n)}} \right| \leqslant (1-\alpha)\chi, \tag{32}$$

where n is the number of the iteration step, j is the number of the calculation point in the quadrature equation used in the numerical integration along the coordinate, and χ is a parameter which takes on values between $5\cdot10^{-4}$ and $5\cdot10^{-6}$, depending on the calculation accuracy required. It is easy to show that under condition (32) the relative discrepancy in terms of $\Delta\theta$ is also smaller than $(1-\alpha)\chi$.

The calculations were carried out on a BÉSM-4 computer with the help of an ALGOL-60 program. The computer time required for the solution of a "gray" problem ranged from 5 (l_x = 14) to 20 (l_x = 30) min and was essentially independent of the number of calculation points along the angle, l_μ. In the numerical solution of "selective" problems, the computer time was either essentially the same as that for a "gray" problem (for small values of the product $l_\nu l_\mu$ or was longer by a factor of about 1.5 (for $l_\nu l_\mu$ = 150). The reason for this difference is that a considerable amount of time is required for exchange with the external memory, where the values of the functions J and G are stored.

To check the accuracy of this calculation method it was necessary to compare the results with those calculated by other investigators (Table 1). This comparison turned out to be possible only for the "gray" problems, since numerical values of k_ν have not been given in papers which take the selectivity into account.

In the calculations leading to the data shown in Table 1 the number of calculation points along x was 30, while along μ it was 8. The parameter χ was 0.00002. Since the problem formulated in the papers used for comparison was slightly different from our own problem (the temperature drop $\Delta\theta$ was specified, and the total flux a was calculated from it), we used the following procedure for the comparison: As the initial values of a (the upper numbers) we adopted those found by the other investigators for radiative–conductive heat transfer described by the parameters shown in parentheses. The results of our calculation are the temperature

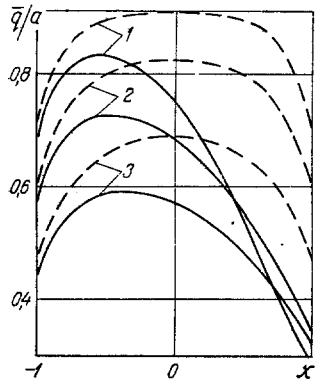

Fig. 2

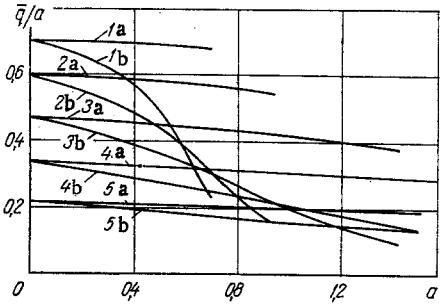

Fig. 3

Fig. 2. Distribution of the radiative flux density in a plane slab (R = 0, τ = 1). Solid curve) a = 0.065; dashed) 0.005. 1) n = 0.5; 2) 1; 3) 2.

Fig. 3. Contribution of the radiative energy flux to the total heat flux as a function of the total energy flux density (R = 0, τ = 1): 1) N = 0.5; 2) 1; 3) 2; 4) 4; 5) 8; a) x = −1; b) 1.

drops (the lower numbers). The same values of $\Delta\theta$ were compared with the values of $\Delta\theta$ shown in parentheses. We see from this table that our data agree best with the data of Crosbie and Viskanta [2], and these authors recommend their results as standards.

The error of the data, due primarily to the discrete formulation of the problem along the coordinate and the angle, was estimated by the method of double calculation; it was found to be lower than 0.05% in all cases. This error increases rapidly with decreasing N, due to the increase in the relative importance of the radiation in the total heat flux. Consequently, the error increases with increasing contribution to the integral terms in Eqs. (16) and (17). For example, with N = 0.1 (R = 0.9, τ = 0.5, a = 0.01) the calculated values of $\Delta\theta$ for the cases of 14, 30, and 44 calculation points are 0.018550, 0.18496, and 0.18499, respectively; for N = 1, these values are 0.0078254, 0.0078287, and 0.0078287.

For a given calculation accuracy, the minimum number of steps along the coordinate leading to the specified accuracy is strongly affected by the distribution of calculation points on the interval (−1, 1). The calculation points are arranged in a symmetric manner with respect to x = 0. Each of the intervals (−1, 0), (0, 1) is generally broken up into three subintervals by the points x = ± 0.95 and ± 0.85. In the subinterval (±1, ± 0.95) the Lobatto quadrature is used in the numerical integration, while in the other two subintervals the Gauss quadratures are used. The average density of calculation points in the first subinterval is higher than that in the second, while that in the second is higher than that in the third.

The model of a gray medium serves as a sort of standard for estimating the accuracy and applicability limits of various numerical methods, but it is also attractive because of the possibility for a substantial reduction in the number of independent parameters determining the radiative−conductive heat transfer. For a plane slab with opaque boundaries, having a single reflection coefficient, which is independent of the angle, the number of such parameters is only four: R, τ, N and a. As a rule, the influence of only the first three of these parameters on the temperature distribution in the slab and on the total heat flux for a given value of $\Delta\theta$ is studied. Here a form of the inverse problem is treated: that of calculating the temperature drops and radiative fluxes in the slab for a given value of a.

We see from Fig. 1a, b that the nonlinearity of the problem due to the radiative flux, which is manifested in a violation of the direct proportionality between $\Delta\theta$ and a, is particularly marked at small values of τ and N and at large values of a. Figure 2 shows the distribution of the radiative flux in the plane slab. Quite naturally, the minimum contribution of the radiation to the total energy flux is found at the cold boundary of the plate. The difference between the values of $\bar{q}/a$ at the hot and cold boundaries falls off gradually with decreasing a. Small values of a correspond to small temperature drops and thus to regimes in which the problem can be treated as linear. In this case the temperature gradients become symmetric with respect to the center of the slab, so that the function $\bar{q}/a$ becomes symmetric.

The contribution of the radiation to the total heat flux increases with decreasing a, as can be seen particularly clearly in Fig. 3. It is also clear from the discussion above why the curves of $\bar{q}/a$ corresponding to x = 1 and x = −1 coincide in the case a = 0.

NOTATION

ξ or x — is the coordinate of the point in the slab ($\in[-1, 1]$);
ψ — is the angle measured from the positive X axis;
$\mu = \cos\psi$;
T — is the temperature;
$\theta = T/T_1$;
Q — is the total energy flux density;
q — is the radiative flux density;
n — is the refractive index;
$\overline{n}^2$ — is the value of n^2 averaged over the spectrum;
k — is the absorption coefficient;
$\varkappa$ — is the absorption coefficient of boundaries;
R — is the reflection coefficient;
Λ — is the thermal conductivities;
h — is the half-thickness of slab;
$\tau = kh$;

$$a = Q/(\overline{n}^2\sigma T_1^4); \quad \overline{q} = q/(\overline{n}^2\sigma T_1^4);$$

$$N = \Lambda/(\overline{n}^2\sigma T_1^3 h); \quad \gamma = \tau/\mu; \quad \beta = 1/[2\gamma(1 - R_1R_2)e^{-4\gamma}];$$

g — is the Green's function (13);
j — is the primitive of Green's function, (20);
J_i, G_i — are the kernels of integral equations (18) and (19);
$\hat{\eta}_i\theta$ — is given by Eq. (21);
$I^{(i)}_{p,\nu}$ — is the i-th derivative of the spectral intensity of the equilibrium radiation in vacuum with respect to the temperature;
m — is the total number of terms in the Taylor-series expansion of $I'_{p,\nu}(T)$;
l — is the number of calculation points of the quadrature equation.

Indices:

x — is the dependence on coordinate;
ν — is the spectral dependence;
μ — is the angular dependence;
1, 2 — are the hot and cold boundaries, respectively;
0 — is a certain temperature near which the Taylor-series expansion is carried out.

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